# Generalized Arithmetic Progressions–A Survey

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Dedicated to my esteemed teacher the Shri. S. Iyer who taught me Euclid, Descartes, Progressions.

**Abstract.** Given any Arithmetic Progression we could treat the Sequence of its Sum upto $n$ terms as another Progression and Find a Formula for the Sum of this Progression, repeat the process as many times as we may desire $\{$find the Super-Sums of the Original Arithmetic Progression upto any [$\alpha$-Level] we may say$\}$ and Find a Formula for the Sum upto $n$ terms in each case. The Progressive Rhythm upto any level [$\alpha$- Level] may be cumilatively collected and A Generalized Arithmetic Progression Defined and its Sum upto $n$ terms and Super-Sums upto $n$ terms may also be formulated analytically.

**Key words:-** Arithmetic Progressions, Infinite Generalizations.

**AMS Subject Classification No:-** 11A25; 11A99.

## 1 Super-sums of an Arithmetic progression

Let $\left[\begin{matrix} \\ \mathbf{A}_n \\ [a,b] \end{matrix}\right] = a+(n-1)b = \begin{matrix} [1] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$ be a given Arithmetic Progression with "$a$" as the Initiating Term and "$b$" as the Basic Difference where "$a$" and "$b$" are algebraic numbers. We can determine the Super-Sums upto any Level $\infty = 1, 2, 3 \cdots \infty$. The ordinary sum of the Arithmetic Progression is clearly the Super-Sum Level–1.

**Notation**

In $[S]_n \left[\begin{matrix} [\alpha] \\ \mathbf{A}_n \\ [a,b] \end{matrix}\right]$, $[S]_n$ is the Sum upto $n$ terms of the $\alpha^{th}$ Level of the Arithmetic Progression [ super-sum $\alpha^{\text{th}}$ Level] and it yields the $\begin{matrix} [\alpha+1] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$ the $(\alpha+1)^{\text{th}}$ Level of the given Arithmetic Progression.

We define Arithmetic Progression.

$$\left[\begin{matrix} \\ \mathbf{A}_n \\ [a,b] \end{matrix}\right] = a + (n-1)b = \begin{matrix} [1] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "$n$" and for the General

Result the induction is performed on “$\alpha$”. The Routine Steps are omitted to save Eternal Space-Time!

$$[S]_n \begin{bmatrix} [1] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = an + \left[ \frac{(n-1)(n)}{2\,!} \right] b = \begin{matrix} [2] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

[Traditional Ancient Result]

$$[S]_n \begin{bmatrix} [2] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a \left[ \frac{n(n+1)}{2} \right] + \left[ \frac{(n-1)(n)(n+1)}{3\,!} \right] b = \begin{matrix} [3] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

$$[S]_n \begin{bmatrix} [3] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a \left[ \frac{n(n+1)(n+2)}{3\,!} \right] +$$

$$\left[ \frac{(n-1)(n)(n+1)(n+2)}{4\,!} \right] b = \begin{matrix} [4] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

$$[S]_n \begin{bmatrix} [4] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a \left[ \frac{n(n+1)(n+2)(n+3)}{4\,!} \right] +$$

$$\left[ \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5\,!} \right] b = \begin{matrix} [5] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

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$$[S]_n \begin{bmatrix} [\alpha-1] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha-2)}{(\alpha-1)\,!}\right] +$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha-2)}{\alpha\,!}\right] b = \begin{matrix} [\alpha] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

$$[S]_n \begin{bmatrix} [\alpha] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha-1)}{\alpha\,!}\right] +$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha-1)}{(\alpha+1)\,!}\right] b = \begin{matrix} [\alpha+1] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

$$[S]_n \begin{bmatrix} [\alpha+1] \\ \mathbf{A}_n \\ [a,b] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha)}{(\alpha+1)\,!}\right] +$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha)}{(\alpha+2)\,!}\right] b = \begin{matrix} [\alpha+2] \\ \mathbf{A}_n \\ [a,b] \end{matrix}$$

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## 2 Inductive-progression

$$\begin{bmatrix} \\ \mathbf{I}_n \\ [a,b,c] \end{bmatrix} = a + (n-1)b + \left[\frac{(n-2)(n-1)}{2}\right] c = \begin{matrix} [1] \\ \mathbf{I}_n \\ [a,b,c] \end{matrix}$$

"$a$" as the Initiating Term and "$b$" as the Basic Difference and "$c$" is the inductive Factor [a,b, and c are algebraic numbers]. We can determine the Super-Sums upto any Level $\alpha = 1, 2, 3$---$\infty$. The ordinary Sum of the INDUCTIVE-PROGRESSION IS clearly the Super-Sum Level - 1.

**Notation**

In $[S]_n \begin{bmatrix} [\alpha] \\ \mathbf{I}_n \\ [a,b,c] \end{bmatrix}$ $[S]_n$ is the Sum upto $n$ terms of the $\alpha^{\text{th}}$ Level of the INDUCTIVE-PROGRESSION [ super-sum $\alpha^{\text{th}}$ Level ] and it yields the $\begin{matrix} [a+1] \\ \mathbf{I}_n \\ [a,b,c] \end{matrix}$ the $(\alpha+1)^{\text{th}}$ Level of the given INDUCTIVE- PROGRESSION.

$$\begin{bmatrix} \\ \mathbf{I}_n \\ [a,b,c] \end{bmatrix} = a + (n-1)b + \left[\frac{(n-2)(n-1)}{2}\right] c = \begin{matrix} [1] \\ \mathbf{I}_n \\ [a,b,c] \end{matrix}$$

{An interesting example of INDUCTIVE-PROGRESSION is the set of Triangular Numbers. [Substitute $a = 1, b = 2, c = 1$].}

Now we can easily prove by Induction that the sum of this progression up to $n$ terms is

$$[S]_n \begin{bmatrix} [1] \\ \mathbf{I}_n \\ [a,b,c] \end{bmatrix} = an + \left[\frac{(n-1)(n)}{2\,!}\right] b + \left[\frac{(n-2)(n-1)(n)}{3\,!}\right]$$

$$c = \begin{matrix} {[2]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "$n$" and for the General Result the induction is performed on "$\alpha$". The Routine Steps are omitted to save Eternal Space-Time!

$$[S]_n \left[ \begin{matrix} {[2]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix} \right] = a \left[ \frac{n(n+1)}{2\,!} \right] + \left[ \frac{(n-1)(n)(n+1)}{3\,!} \right] b$$

$$+ \left[ \frac{(n-2)(n-1)(n)(n+1)}{4\,!} \right] c = \begin{matrix} {[3]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix}$$

$$[S]_n \left[ \begin{matrix} {[3]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix} \right] = a \left[ \frac{n(n+1)(n+2)}{3\,!} \right] + \left[ \frac{(n-1)(n)(n+1)(n+2)}{4\,!} \right] b$$

$$+ \left[ \frac{(n-2)(n-1)(n)(n+1)(n+2)}{5\,!} \right] c = \begin{matrix} {[4]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix}$$

$$[S]_n \left[ \begin{matrix} {[4]} \\ \mathbf{I}_n \\ {[a,b,c]} \end{matrix} \right] = a \left[ \frac{n(n+1)(n+2)(n+3)}{4\,!} \right]$$

$$+\left[\frac{(n-1)(n)(n+1)(n+2)(n+3)}{5\,!}\right] b$$

$$+\left[\frac{(n-2)(n-1)(n)(n+1)(n+2)(n+3)}{6\,!}\right] c = \begin{matrix}[5]\\ \mathbf{I}_n\\ [a,b,c]\end{matrix}$$

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$$[S]_n\begin{bmatrix}[\alpha-1]\\ \mathbf{I}_n\\ [a,b,c]\end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha-2)}{(\alpha-1)\,!}\right]+$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha-2)}{\alpha\,!}\right]$$

$$b+\left[\frac{(n-2)(n-1)(n)\text{---}(n+\alpha-2)}{(\alpha+1)\,!}\right] c = \begin{matrix}[\alpha]\\ \mathbf{I}_n\\ [a,b,c]\end{matrix}$$

$$[S]_n\begin{bmatrix}[\alpha]\\ \mathbf{I}_n\\ [a,b,c]\end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha-1)}{\alpha\,!}\right]+$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha-1)}{(\alpha+1)\,!}\right]$$

$$b+\left[\frac{(n-2)(n-1)(n)\text{---}(n+\alpha-1)}{(\alpha+2)\,!}\right] c = \begin{matrix}[\alpha+1]\\ \mathbf{I}_n\\ [a,b,c]\end{matrix}$$

$$[S]_n\begin{bmatrix}[\alpha+1]\\ \mathbf{I}_n\\ [a,b,c]\end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{------}(n+\alpha)}{(\alpha+1)\,!}\right]+$$

$$\left[\frac{(n-1)(n)(n+1)\text{------}(n+\alpha)}{(\alpha+2)\,!}\right]$$

$$b+\left[\frac{(n-2)(n-1)(n)\text{---}(n+\alpha)}{(\alpha+3)\,!}\right]c=\underset{[a,b]}{\overset{[\alpha+2]}{\mathbf{I}_n}}$$

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## 3 Generalized Arithmetic Progressions-simple [* Arithmetic Progression]

We define the
GENERALIZED ARITHMETIC PROGRESSIONS-
SIMPLE [* ARITHMETIC PROGRESSION]

$$\left[\begin{matrix}\overset{*}{A}_n\\ [a,b_1,\text{---},b_\beta]\end{matrix}\right]=a+(n-1)b_1+\left[\frac{(n-1)(n)}{2}\right]b_2+$$

$$\left[\frac{(n-1)(n)(n+1)}{3\,!}\right]b_3+\text{---}+$$

$$+\left[\frac{(n-1)(n)(n+1)\text{---}(n+\beta-2)}{\beta\,!}\right]b_\beta$$

$$=a+\sum_{\phi=1}^{\beta}\left[\frac{(n-1)\text{---}(n+\phi-2)}{\phi\,!}\right]b_\phi=\underset{[a,b_1,\text{---},b_\beta]}{\overset{[1]}{\overset{*}{\mathbf{A}}_n}}$$

“$a$” as the initiating Term and “$b_1$”, “$b_2$”, “$b_3$” ---“$b_\beta$” as the Basic Differences [$a, b_1, b_2, b_3,$ ---, $b_\beta$ are algebraic numbers ]. We can determine the Super-Sums of the GENERALIZED ARITHMETIC PROGRESSION -SIMPLE UPTO any Level $\alpha = 1, 2, 3$ ---$\infty$. The ordinary

Sum of GENERALIZED ARITHMETIC PROGRESSIO -SIMPLE is clearly the Super-Sum Level-1 of the same. "$\beta$" may be called Generalization of the GENERALIZED ARITHMETIC PROGRESSION - SIMPLE.

**Notation**

In $[S]_n \left[\begin{matrix} [\alpha] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{matrix}\right]$ $[S]_n$ is the Sum upto $n$ terms of the $\alpha^{\text{th}}$ Level of the GENERALIZED ARITHMETIC PROGRESSION-SIMPLE [Super-Sum $\alpha^{\text{th}}$ Level ] and it yields the $\begin{matrix} [a+1] \\ \mathbf{I}_n \\ [a, b, c] \end{matrix}$ the $(\alpha+1)^{\text{th}}$ Level of the given GENERALIZED ARITHMETIC PROGRESSION-SIMPLE.

$$\left[\begin{matrix} * \\ \mathbf{A}_n \\ [a, b_1, \text{---}, b_\beta] \end{matrix}\right] = a + \sum_{\phi=1}^{\beta} \left[\frac{(n-1)\text{---}(n+\phi-2)}{\phi!}\right] b_\phi$$

$$= \left[\begin{matrix} [\alpha] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{matrix}\right]$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "$n$"and for The General Result the induction is performed on "$\alpha$". The Routine Steps are omitted to save Eternal Space -Time!

$$
\begin{aligned}
[S]n \begin{bmatrix} [1] \\ \mathbf{A}_n^* \\ [a, b_1, \text{- - -}, b_\beta] \end{bmatrix} &= an + \left[\frac{(n-1)(n)}{2!}\right] b_1 \\
&+ \left[\frac{(n-1)(n)(n+1)}{3!}\right] b_2 + \text{- - -}+ \\
&+ \left[\frac{(n-1)(n)(n+1)\text{- - -}(n+\beta-1)}{(\beta+1)!}\right] b_\beta \\
&= an + \sum_{\phi=1}^{\beta} \left[\frac{(n-1)\text{- - -}(n+\phi-1)}{(\phi+1)}(\phi+1)!\right] b_\phi \\
&= \begin{bmatrix} [2] \\ \mathbf{A}_n^* \\ [a, b_1, \text{- - -}, b_\beta] \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
[S]n \begin{bmatrix} [2] \\ \mathbf{A}_n^* \\ [a, b_1, \text{- - -}, b_\beta] \end{bmatrix} &= a\left[\frac{n(n+1)}{2!}\right] + \sum_{\phi=1}^{\beta} \left[\frac{(n-1)\text{- - -}(n+\phi)}{(\phi+2)!}\right] b_\phi \\
&= \begin{matrix} [3] \\ \mathbf{A}_n^* \\ [a, b_1, \text{- - -}, b_\beta] \end{matrix}
\end{aligned}
$$

$$[S]n \begin{bmatrix} [3] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)}{3!}\right] + \sum_{\phi=1}^{\beta}\left[\frac{(n-1)\text{---}(n+\phi+1)}{(\phi+3)!}\right] b_\phi = \begin{matrix} [4] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{matrix}$$

$$[S]n \begin{bmatrix} [4] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)(n+3)}{4!}\right] + \sum_{\phi=1}^{\beta}\left[\frac{(n-1)\text{---}(n+\phi+2)}{(\phi+4)!}\right] b_\phi = \begin{matrix} [5] \\ \mathbf{A}^{*}_{n} \\ [a, b_1, \text{---}, b_\beta] \end{matrix}$$

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$$[S]n \begin{bmatrix} [\alpha - 1] \\ \mathbf{A}_n^* \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix} = a \left[ \frac{n(n+1)(n+2)\text{---}(n+\alpha-2)}{(\alpha-1)!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-1)\text{---}(n+\phi+\alpha-3)}{(\phi+\alpha-1)!} \right] b_\phi = \begin{matrix} [\alpha] \\ \mathbf{A}_n^* \\ [a, b_1, \text{---}, b_\beta] \end{matrix}$$

$$[S]n \begin{bmatrix} [\alpha] \\ \mathbf{A}_n^* \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix} = a \left[ \frac{n(n+1)(n+2)\text{---}(n+\alpha-1)}{\alpha!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-1)\text{---}(n+\phi+\alpha-2)}{(\phi+\alpha)!} \right] b_\phi = \begin{matrix} [\alpha+1] \\ \mathbf{A}_n^* \\ [a, b_1, \text{---}, b_\beta] \end{matrix}$$

$$[S]n \begin{bmatrix} [\alpha+1] \\ \mathbf{A}_{n}^{*} \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix} = a\left[\frac{n(n+1)(n+2)\text{---}(n+\alpha)}{(\alpha+1)!}\right] + \sum_{\phi=1}^{\beta}\left[\frac{(n-1)\text{---}(n+\phi+\alpha-1)}{(\phi+\alpha+1)!}\right] b_\phi = \begin{matrix} [\alpha+2] \\ \mathbf{A}_{n}^{*} \\ [a, b_1, \text{---}, b_\beta] \end{matrix}$$

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## 4 Generalized Arithmetic Progressions-Random [$*^r$ Arithmetic Progressions]

We define the
GENERALIZED ARITHMETIC PROGRESSIONS -RANDOM
[$*^r$ ARITHMETIC PROGRESSIONS]

$$\underset{[@, j_1, \text{---}, j_\beta]}{\begin{bmatrix} \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{---}, b_\beta] \end{bmatrix}} = a + (n-j_1)b_1 + \left[\frac{(n-j_2)(n-j_2+1)}{2!}\right] b_2 + \left[\frac{(n-j_3)(n-j_3+1)(n-j_3+2)}{3!}\right] b_3 + \text{---} + \left[\frac{(n-j_\beta)(n-j_\beta+1)(n-j_\beta+2)\text{---}(n-j_\beta+\beta-1)}{\beta!}\right] b_\beta$$

$$= a + \sum_{\phi=1}^{\beta} \left[ \frac{(n-j_\phi)\text{---}(n-j_\phi+\phi-1)}{\phi!} \right] b_\phi$$

$$= \begin{matrix} [1] \\ \mathbf{A}_n^{*r} \\ [a, b_1, \text{---}, b_\beta] \\ [@, j_1, \text{---}, j_\beta] \end{matrix}$$

with "$a$" as the Initiating Term and "$b_1$", "$b_2$","$b_3$" --- "$b_\beta$" as the Basic Differences $[a, b_1, b_2, b_3, \text{---}, b_\beta$ are algebraic numbers ] induced at [@] "$j_1$", "$j_2$", "$j_2$", ---,"$j_\beta$" respectively $[j_i > 1,\ i = 1, 2\text{---}\beta]$. We can determine the Super-Sums of the GENERALIZED ARITHMETIC PROGRESSION-RANDOM upto any Level $\alpha = 1, 2, \text{---}\infty$. The ordinary Sum of GENERALIZED ARITHMETIC PROGRESSION -RANDOM is clearly the Super- Sum Level -1 of the same. "$\beta$" may be called Generalization- Level of the GENERALIZED ARITHMETIC PROGRESSION -RANDOM. $J_i, (I = 1, 2\text{---}\beta)$ may be called the Inductive Points of the GENERALIZED ARITHMETIC PROGRESSION -RANDOM.

**Notation**

In $[S]_n \left[ \begin{matrix} [\alpha] \\ \mathbf{A}_n^{*r} \\ [a, b_1, \text{---}, b_\beta] \end{matrix} \right]_{[@\, j_1, \text{---}, j_\beta]}$, $[S]_n$ is the sum upto $n$ terms of the $\alpha^{\text{th}}$ Level

of the GENERALIZED ARITHMETIC PROGRESSION-RANDOM [Super-

Sum $\alpha^{\text{th}}$ Level] and it yields the $\begin{matrix}[a+1]\\ \mathbf{A}_{n}^{*r}\\ [a,b_1,\text{---},b_\beta]\\ [@j_1,\text{---},j_\beta]\end{matrix}$ the $(\alpha+1)^{\text{th}}$ Level of the given GENERALIZED ARITHMETIC PROGRESSION-RANDOM.

$$\begin{bmatrix} \mathbf{A}_{n}^{*r}\\ [a,b_1,\text{---},b_\beta]\end{bmatrix}_{[@,j_1,\text{---},j_\beta]} = a + \sum_{\phi=1}^{\beta}\left[\frac{(n-j_\phi)\text{---}(n-j_\phi+\phi-1)}{\phi!}\right]b_\phi$$

$$= \begin{matrix}[1]\\ \mathbf{A}_{n}^{*r}\\ [a,b_1,\text{---},b_\beta]\\ [@j_1,\text{---},j_\beta]\end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "$n$" and for The General Result the induction is performed on "$\alpha$". The Routine Steps are omitted to save Eternal Space-Time!

$$\begin{bmatrix}[1]\\ \mathbf{A}_{n}^{*r}\\ [a,b_1,\text{---},b_\beta]\end{bmatrix}_{[@,j_1,\text{---},j_\beta]} = an + \sum_{\phi=1}^{\beta}\left[\frac{(n-j_\phi)\text{---}(n-j_\phi+\phi)}{(\phi+1)!}\right]b_\phi$$

$$= \begin{array}{c} {[2]} \\ \mathbf{A}_n^{*r} \\ {[a, b_1, ---, b_\beta]} \\ {[@j_1, ---, j_\beta]} \end{array}$$

$$[S]_n \left[ \begin{array}{c} {[2]} \\ \mathbf{A}_n^{*r} \\ {[a, b_1, ---, b_\beta]} \end{array} \right]_{[@, j_1, ---, j_\beta]} = a \left[ \frac{n(n+1)}{2!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-j_\phi)---(n-j_\phi+\phi+1)}{(\phi+2)!} \right] b_\phi$$

$$= \begin{array}{c} {[3]} \\ \mathbf{A}_n^{*r} \\ {[a, b_1, ---, b_\beta]} \\ {[@j_1, ---, j_\beta]} \end{array}$$

$$[S]_n \left[ \begin{array}{c} {[3]} \\ \mathbf{A}_n^{*r} \\ {[a, b_1, ---, b_\beta]} \end{array} \right]_{[@, j_1, ---, j_\beta]} = a \left[ \frac{n(n+1)(n+2)}{3!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-j_\phi)---(n-j_\phi+\phi+2)}{(\phi+3)!} \right] b_\phi$$

$$= \begin{matrix} [4] \\ \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{---}, b_\beta] \\ [@j_1, \text{---}, j_\beta] \end{matrix}$$

$$[S]_n \left[ \begin{matrix} [4] \\ \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{---}, b_\beta] \end{matrix} \right]_{[@, j_1, \text{---}, j_\beta]} = a \left[ \frac{n(n+1)(n+2)(n+3)}{4!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-j_\phi)\text{---}(n-j_\phi+\phi+3)}{(\phi+4)!} \right] b_\phi$$

$$= \begin{matrix} [5] \\ \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{---}, b_\beta] \\ [@j_1, \text{---}, j_\beta] \end{matrix}$$

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$$[S]_n \left[ \begin{matrix} [\alpha-1] \\ \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{---}, b_\beta] \end{matrix} \right]_{[@, j_1, \text{---}, j_\beta]} = a \left[ \frac{n(n+1)(n+2)\text{---}(n+\alpha-2)}{(\alpha-1)!} \right] + \sum_{\phi=1}^{\beta} \left[ \frac{(n-j_\phi)\text{---}(n-j_\phi+\phi+\alpha-2)}{(\phi+\alpha-1)!} \right] b_\phi$$

$$= \begin{matrix}[\alpha]\\ \mathbf{A}_{n}^{*r}\\ [a, b_1, ---, b_\beta]\\ [@j_1, ---, j_\beta]\end{matrix}$$

$$[S]n \begin{bmatrix}[\alpha]\\ \mathbf{A}_{n}^{*r}\\ [a, b_1, ---, b_\beta]\end{bmatrix} = a\left[\frac{n(n+1)(n+2)---(n+\alpha-1)}{\alpha!}\right]$$

$$[@, j_1, ---, j_\beta]$$

$$+\sum_{\phi=1}^{\beta}\left[\frac{(n-j_\phi)---(n-j_\phi+\phi+\alpha-1)}{(\phi+\alpha)!}\right] b_\phi$$

$$= \begin{matrix}[\alpha+1]\\ \mathbf{A}_{n}^{*r}\\ [a, b_1, ---, b_\beta]\\ [@j_1, ---, j_\beta]\end{matrix}$$

$$[S]n \begin{bmatrix}[\alpha+1]\\ \mathbf{A}_{n}^{*r}\\ [a, b_1, ---, b_\beta]\end{bmatrix} = a\left[\frac{n(n+1)(n+2)---(n+\alpha)}{(\alpha+1)}\right]$$

$$[@, j_1, ---, j_\beta]$$

$$+\sum_{\phi=1}^{\beta}\left[\frac{(n-j_\phi)---(n-j_\phi+\phi+\alpha+1)}{(\phi+\alpha+1)!}\right] b_\phi$$

$$= \begin{array}{c} [\alpha + 2] \\ \mathbf{A}_{n}^{*r} \\ [a, b_1, \text{-\,-\,-}, b_\beta] \\ [@ j_1, \text{-\,-\,-}, j_\beta] \end{array}$$

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We could of course induct and repeat the same inductive-block at as many random points of entry and generalize appropriately. The details though trivial are cumbersome. [See:1]

Since the Initiating term and Basic differences can be any algebraic number, we can see that each family of the Progressions elucidated here, defines a unique Algebraic Field of Sequences.

## REFERENCES

[1] Narayanan Raghunathan: *Functions and their Progressions- An Elementary Text*. [unpublished]