OM SHRI MAHA GANAPATHAYE NAMA:

OM POORNAMADA: POORNAMIDAM
POORNAT POORNAMUDACHYATE
POORNASYA POORNAMADAYA POORNAMEVAVASHISHYATE

OM SHOONYAMADA: SHOONYAMIDAM
SHOONYAT SHOONYAMUDACHYATE
SHOONYASYA SHOONYAMADAYA
SHOONYAMEVAVASHISHYATE

GENERALIZED GEOMETRIC PROGRESSIONS – A SURVEY

Dedicated to my esteemed teacher Shri. C. G. George
who taught me Algebra, Calculus (SSKZM)

Narayanan Raghunathan

Abstract :

Given any Geometric Progression we could treat the Sequence of its Sum upto n terms as another Progression and Find a Formula for the Sum of this Progression, repeat the process as many times as we may desire {find the Super–Sums of the Original Geometric Progression upto any [α–Level] we may say} and Find a Formula for the Sum upto n terms in each case. The Progressive Rhythm upto any level [α–Level] may be cumilatively collected and A Generalized Geometric Progression Defined and its Sum upto n terms and Super–Sums upto n terms may also be formulated analytically.

1] SUPER–SUMS OF A GEOMETRIC PROGRESSION

Let $\begin{bmatrix} \\ G_n \\ [a,r] \end{bmatrix} = ar^{(n-1)} = \begin{matrix} [1] \\ G_n \\ [a,r] \end{matrix}$ $[n \geq 1]$ be a given Geometric Progression with "a" as the Initiating Term and "r" as the Common Ratio where "a" and "r" are algebraic numbers ($r \neq 1$). We can determine the Super–Sums upto any Level $\alpha = 1, 2, 3 \text{ --- } \infty$. The ordinary Sum of the Geometric Progression is clearly the Super–Sum Level–1.

Notation

In $[S]_n \begin{bmatrix} [\alpha] \\ G_n \\ [a,r] \end{bmatrix}$, $[S]_n$ is the Sum upto n terms of the α^{th} Level of the Geometric Progression [Super–Sum α^{th} Level] and it yields the $\begin{bmatrix} [\alpha+1] \\ G_n \\ [a,r] \end{bmatrix}$ the $(\alpha+1)^{th}$ Level of the given Geometric Progression.

We define the Geometric Progression.

$$\begin{bmatrix} \\ G_n \\ [a,r] \end{bmatrix} = ar^{(n-1)} = \begin{matrix} [1] \\ G_n \\ [a,r] \end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "n" and for The General Result the induction is performed on "α". The Routine Steps are omitted to save Eternal Space–Time !

$$[S]_n \begin{bmatrix} [1] \\ G_n \\ [a,r] \end{bmatrix} = \frac{a(r^n - 1)}{r - 1} = \begin{matrix} [2] \\ G_n \\ [a,r] \end{matrix} \qquad \text{[Traditional Ancient Result]}$$

$$[S]_n \begin{bmatrix} [2] \\ G_n \\ [a,r] \end{bmatrix} = a\Bigg[\frac{\frac{(r^{(n+1)} - 1)}{(r-1)} - n}{(r-1)}\Bigg] = a\Bigg[\frac{r^{(n+1)} - 1 - n(r-1)}{(r-1)^2}\Bigg]$$

$$= a\Big[\frac{r^{(n+1)} - [n(r-1)+1]}{(r-1)^2}\Big] = a\Big[\frac{r^{(n+1)} - \dfrac{[n^2(r-1)^2-1]}{[n-(r-1)-1]}}{(r-1)^2}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^2(r-1)^2-1]}{(r-1)^2[n(r-1)-1]}\Big] = \begin{matrix}[3]\\ G_n\\ [a,r]\end{matrix}$$

$$[S]_n \begin{bmatrix}[3]\\ G_n\\ [a,r]\end{bmatrix} = a\Big[\frac{r^{(n+2)} - [n^2(r-1)^2+n(r-1)+1]}{(r-1)^3}\Big] = a\Big[\frac{r^{(n+1)} - \dfrac{[n^3(r-1)^3-1]}{[n-(r-1)-1]}}{(r-1)^3}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n-(r-1)-1] - [n^3(r-1)^3-1]}{(r-1)^3[n(r-1)-1]}\Big] = \begin{matrix}[4]\\ G_n\\ [a,r]\end{matrix}$$

$$[S]_n \begin{bmatrix}[4]\\ G_n\\ [a,r]\end{bmatrix} = a\Big[\frac{r^{(n+3)} - [n^3(r-1)^3+n^2(r-1)^2+n(r-1)+1]}{(r-1)^4}\Big] =$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^4(r-1)^4-1]}{[n-(r-1)-1]}}{(r-1)^4}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^4(r-1)^4-1]}{(r-1)^4[n(r-1)-1]}\Big] = \begin{matrix}[5]\\ G_n\\ [a,r]\end{matrix}$$

$$[S]_n \begin{bmatrix}[5]\\ G_n\\ [a,r]\end{bmatrix} = a\Big[\frac{r^{(n+4)} - [n^4(r-1)^4+n^3(r-1)^3+n^2(r-1)^2+n(r-1)+1]}{(r-1)^5}\Big] =$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^5(r-1)^5-1]}{[n(r-1)-1]}}{(r-1)^5}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^5(r-1)^5-1]}{(r-1)^5[n(r-1)-1]}\Big] = \begin{matrix}[6]\\ G_n\\ [a,r]\end{matrix}$$

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$$[S]_n \begin{bmatrix} [\alpha-1] \\ G_n \\ [a,r] \end{bmatrix} = a\Big[\frac{r^{(n+\alpha-2)} - [n^{(\alpha-2)}(r-1)^{(\alpha-2)} + \text{---} + n^2(r-1)^2 + n(r-1)+1]}{(r-1)^{(\alpha-1)}}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^{(\alpha-1)}(r-1)^{(\alpha-1)} - 1]}{[n(r-1)-1]}}{(r-1)^{(\alpha-1)}}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{(\alpha-1)}(r-1)^{(\alpha-1)} - 1]}{(r-1)^{(\alpha-1)}[n(r-1)-1]} = \begin{matrix} [\alpha] \\ G_n \\ [a,r] \end{matrix}$$

$$[S]_n \begin{bmatrix} [\alpha] \\ G_n \\ [a,r] \end{bmatrix} = a\Big[\frac{r^{(n+\alpha-1)} - [n^{(\alpha-1)}(r-1)^{(\alpha-1)} + \text{---} + n^2(r-1)^2 + n(r-1)+1]}{(r-1)^{\alpha}}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^{\alpha}(r-1)^{\alpha} - 1]}{[n(r-1)-1]}}{(r-1)^{\alpha}}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{\alpha}(r-1)^{\alpha} - 1]}{(r-1)^{\alpha}[n(r-1)-1]}\Big] = \begin{matrix} [\alpha+1] \\ G_n \\ [a,r] \end{matrix}$$

$$[S]_n \begin{bmatrix} [\alpha+1] \\ G_n \\ [a,r] \end{bmatrix} = a\Big[\frac{r^{(n+\alpha)} - [n^{\alpha}(r-1)^{\alpha} + \text{---} + n^2(r-1)^2 + n(r-1)+1]}{(r-1)^{(\alpha+1)}}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^{(\alpha+1)}(r-1)^{(\alpha+1)} - 1]}{[n(r-1)-1]}}{(r-1)^{(\alpha+1)}}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{(\alpha+1)}(r-1)^{(\alpha+1)} - 1]}{(r-1)^{(\alpha+1)}[n(r-1)-1]}\Big] = \begin{matrix} [\alpha+2] \\ G_n \\ [a,r] \end{matrix}$$

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2] INDUCTIVE GEOMETRIC PROGRESSIONS

We define the INDUCTIVE–GEOMETRIC PROGRESSION

$$\begin{bmatrix} \\ GI_n \\ [a,r,s] \end{bmatrix} = ar^{(n-1)} s^{\left(\frac{(n-2)(n-1)}{2!}\right)} = \begin{matrix} [1] \\ GI_n \\ [a,r,s] \end{matrix} \qquad [n \geq 1]$$

"a" as the Initiating Term and "r" as the Common Ratio and "s" is the Inductive Ratio [a, r, and s are algebraic numbers ($r \neq 1$]. We can determine the Super–Sums upto any Level $\alpha = 1, 2, 3 \text{ - - - } \infty$. The ordinary Sum of the INDUCTIVE–GEOMETRIC PROGRESSION is clearly the Super–Sum Level – 1.

Notation

In $[S]_n \begin{bmatrix} [\alpha] \\ GI_n \\ [a,r,s] \end{bmatrix}$, $[S]_n$ is the Sum upto n terms of the α^{th} Level of the INDUCTIVE–GEOMETRIC PROGRESSION [Super–Sum α^{th} Level] and it yields the $\begin{bmatrix} [\alpha+1] \\ GI_n \\ [a,r,s] \end{bmatrix}$ the $(\alpha+1)^{th}$ Level of the given INDUCTIVE–GEOMETRIC PROGRESSION

$$\begin{bmatrix} \\ GI_n \\ [a,r,s] \end{bmatrix} = ar^{(n-1)} s^{\left(\frac{(n-2)(n-1)}{2!}\right)} = \begin{matrix} [1] \\ GI_n \\ [a,r,s] \end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "n" and for The General Result the induction is performed on "α". The Routine Steps are omitted to save Eternal Space–Time!

$$[S]_n \begin{bmatrix} [1] \\ GI_n \\ [a,r,s] \end{bmatrix} = \frac{a(r^n-1)}{(r-1)} \Big[\sum_{i=2}^{(n-1)} (n-i) s^{\left(\frac{(n-i)(n-i+1)}{2!}\right)} \Big] = \begin{matrix} [2] \\ GI_n \\ [a,r,s] \end{matrix}$$

$$[S]_n \begin{bmatrix} [1] \\ GI_n \\ [a,r,s] \end{bmatrix} =$$

$$a\Big[\frac{\dfrac{(r^{(n+1)}-1)}{(r-1)}-n}{(r-1)}\Big]\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)}{2!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}-1-n(r-1)}{(r-1)^2}\Big]\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)}{2!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}-[n(r-1)+1]}{(r-1)^2}\Big]\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)}{2!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}-\dfrac{[n^2(r-1)^2-1]}{[n(r-1)-1]}}{(r-1)^2}\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)}{2!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^2(r-1)^2-1]}{(r-1)^2[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)}{2!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= \begin{matrix} [3] \\ GI_n \\ [a,r,s] \end{matrix}$$

$$[S]_n \begin{bmatrix} [3] \\ GI_n \\ [a,r,s] \end{bmatrix}$$

$$= a\Big[\frac{r^{(n+2)}-[n^2(r-1)^2+n(r-1)+1]}{(r-1)^3}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)}{3!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^3(r-1)^3 - 1]}{[n(r-1)-1]}}{(r-1)^3}\Big]\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)}{3!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^3(r-1)^3 - 1]}{(r-1)^3[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)}{3!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= \begin{matrix}[4]\\ GI_n\\ [a,r,s]\end{matrix}$$

$$[S]_n\begin{bmatrix}[4]\\ GI_n\\ [a,r,s]\end{bmatrix}$$

$$= a\Big[\frac{r^{(n+3)} - [n^3(r-1)^3 + n^2(r-1)^2 + n(r-1) + 1]}{(r-1)^4}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)}{4!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^4(r-1)^4 - 1]}{[n(r-1)-1]}}{(r-1)^4}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)}{4!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^4(r-1)^4 - 1]}{(r-1)^4[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)}{4!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= \begin{matrix}[5]\\ GI_n\\ [a,r,s]\end{matrix}$$

$$[S]_n \begin{bmatrix}[5]\\ GI_n\\ [a,r,s]\end{bmatrix}$$

$$= a\Big[\frac{r^{(n+4)} - [n^4(r-1)^4 + n^3(r-1)^3 + n^2(r-1)^2 + n(r-1) + 1]}{(r-1)^5}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)} \Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)(n-i+4)}{5!}\Big) s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \frac{[n^5(r-1)^5 - 1]}{[n(r-1)-1]}}{(r-1)^5}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)} \Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)(n-i+4)}{5!}\Big) s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^5(r-1)^5 - 1]}{(r-1)^5[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)} \Big(\frac{(n-i)(n-i+1)(n-i+2)(n-i+3)(n-i+4)}{5!}\Big) s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= \begin{matrix}[6]\\ GI_n\\ [a,r,s]\end{matrix}$$

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$$[S]_n \begin{bmatrix}[\alpha-1]\\ GI_n\\ [a,r,s]\end{bmatrix}$$

$$= a\Big[\frac{r^{(n+\alpha-2)} - [n^{(\alpha-2)}(r-1)^{(\alpha-2)} + \text{---} + n^2(r-1)^2 + n(r-1) + 1]}{(r-1)^{(\alpha-1)}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-2)}{(\alpha-1)!}\Big)s\Big(\frac{(n-i)(n-i+1)}{2!}\Big)\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^{(\alpha-1)}(r-1)^{(\alpha-1)} - 1]}{[n(r-1)-1]}}{(r-1)^{(\alpha-1)}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-2)}{(\alpha-1)!}\Big)s\Big(\frac{(n-i)(n-i+1)}{2!}\Big)\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{(\alpha-1)}(r-1)^{(\alpha-1)} - 1]}{(r-1)^{(\alpha-1)}[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-2)}{(\alpha-1)!}\Big)s\Big(\frac{(n-i)(n-i+1)}{2!}\Big)\Big] = \begin{matrix}[\alpha]\\ GI_n\\ [a,r,s]\end{matrix}$$

$$[S]_n\begin{bmatrix}[\alpha]\\ GI_n\\ [a,r,s]\end{bmatrix}$$

$$= a\Big[\frac{r^{(n+\alpha-1)} - [n^{(\alpha-1)}(r-1)^{(\alpha-1)} + \text{---} + n^2(r-1)^2 + n(r-1) + 1]}{(r-1)^{\alpha}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-1)}{\alpha!}\Big)s\Big(\frac{(n-i)(n-i+1)}{2!}\Big)\Big]$$

$$= a\Big[\frac{r^{(n+1)} - \dfrac{[n^{\alpha}(r-1)^{\alpha} - 1]}{[n(r-1)-1]}}{(r-1)^{\alpha}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-1)}{\alpha!}\Big)s\Big(\frac{(n-i)(n-i+1)}{2!}\Big)\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{\alpha}(r-1)^{\alpha} - 1]}{(r-1)^{\alpha}[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha-1)}{\alpha!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big] = \begin{matrix}[\alpha+1]\\ GI_n\\ [a,r,s]\end{matrix}$$

$$[S]_n\begin{bmatrix}[\alpha+1]\\ GI_n\\ [a,r,s]\end{bmatrix}$$

$$= a\Big[\frac{r^{(n+\alpha)}-[n^{\alpha}(r-1)^{\alpha}+\text{---}+n^2(r-1)^2+n(r-1)+1]}{(r-1)^{(\alpha+1)}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha)}{(\alpha+1)!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}-\frac{[n^{(\alpha+1)}(r-1)^{(\alpha+1)}-1]}{[n(r-1)-1]}}{(r-1)^{(\alpha+1)}}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha)}{(\alpha+1)!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^{(\alpha+1)}(r-1)^{(\alpha+1)}-1]}{(r-1)^{(\alpha+1)}[n(r-1)-1]}\Big]$$

$$\Big[\sum_{i=2}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\alpha)}{(\alpha+1)!}\Big)s^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}\Big] = \begin{matrix}[\alpha+2]\\ GI_n\\ [a,r,s]\end{matrix}$$

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3] GENERALIZED GEOMETRIC PROGRESSIONS–SIMPLE [$*$ GEOMETRIC PROGRESSIONS]

We define the
GENERALIZED GEOMETRIC PROGRESSION–SIMPLE [$*$ GEOMETRIC PROGRESSION]

$$\begin{bmatrix} G_n^{*}\\ [a,r,s_1,\text{---}s_\beta]\end{bmatrix} = ar^{(n-1)}s_1^{\Big(\frac{(n-2)(n-1)}{2!}\Big)}\sum_{i=3}^{(n-1)}(n-i)s_2^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$\sum_{i=4}^{(n-1)} \Big(\frac{(n-i)(n-i+1)}{2!}\Big) s_3^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)} \text{- - - - -}$$

$$\sum_{i=\beta+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{- - -}(n-i+\beta-2)}{(\beta-1)!}\Big) s_\beta^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= ar^{(n-1)} \sum_{\phi=1}^{\beta} \sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{- - -}(n-i+\phi-2)}{(\phi-1)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \underset{[a,r,s_1,\text{- - -},s_\beta]}{\overset{[1]}{G_n^*}} \qquad [n \geq 1]$$

"a" as the Initiating Term, "r" as the Common Ratio and "s_1", "s_2", "s_3", - - -, "s_β" as the Inductive Ratios [$a, r, s_1, s_2, s_3,$ - - - $, s_\beta$ are algebraic numbers]. We can determine the Super–Sums of the GENERALIZED GEOMETRIC PROGRESSION–SIMPLE upto any Level $\alpha = 1, 2, 3\text{- - -}\infty$. The ordinary Sum of GENERALIZED GEOMETRIC PROGRESSION – SIMPLE is clearly the Super–Sum Level −1 of the same "β" may be called Generalization–Level of the GENERALIZED GEOMETRIC PROGRESSION –SIMPLE.

Notation

In $[S]_n \left[\underset{[a,r,s_1,\text{- - -},s_\beta]}{\overset{[\alpha]}{G_n^*}}\right]$, $[S]_n$ is the Sum upto n terms of the α^{th} Level of the GENERALIZED GEOMETRIC PROGRESSION –SIMPLE [Super–Sum α^{th} Level] and it yields the $\underset{[a,r,s_1,\text{- - -},s_\beta]}{\overset{[\alpha+1]}{G_n^*}}$ the $(\alpha+1)^{th}$ Level of the given GENERALIZED GEOMETRIC PROGRESSION –SIMPLE.

$$\left[\underset{[a,r,s_1,\text{- - -},s_\beta]}{G_n^*}\right]$$

$$= ar^{(n-1)} \sum_{\phi=1}^{\beta} \sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi-2)}{(\phi-1)!} s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!} \Big)}$$

$$= \begin{matrix} [1] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

ALL the following Sequence of Formulae could be easily proved by the method of Mathematical induction by now traditionally formalized. For each Level, the induction is performed on "n" and for The General Result the induction is performed on "α". The Routine Steps are omitted to save Eternal Space–Time!

$$[S]_n \left[\begin{matrix} [1] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix} \right] =$$

$$= \frac{a(r^n - 1)}{(r-1)} \sum_{\phi=1}^{\beta} \sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi-1)}{\phi!} \Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!} \Big)}$$

$$= \left[\begin{matrix} [2] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix} \right]$$

$$[S]_n \left[\begin{matrix} [2] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix} \right] =$$

$$a \Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^2(r-1)^2 - 1]}{(r-1)^2[n(r-1)-1]} \Big]$$

$$\sum_{\phi=1}^{\beta} \sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi)}{(\phi+1)!} \Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!} \Big)}$$

$$= \begin{matrix} [3] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

$$[S]_n \left[\begin{matrix} [3] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}\right] =$$

$$a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^3(r-1)^3 - 1]}{(r-1)^3[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+1)}{(\phi+2)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [4] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

$$[S]_n \left[\begin{matrix} [4] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}\right] = a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^4(r-1)^4 - 1]}{(r-1)^4[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+2)}{(\phi+3)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [5] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

$$[S]_n \left[\begin{matrix} [5] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}\right] = a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^5(r-1)^5 - 1]}{(r-1)^5[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+3)}{(\phi+4)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [6] \\ G_n^* \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

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$$[S]_n \left[\begin{matrix}[\alpha-1]\\ G_n^*\\ [a,r,s_1,\text{---},s_\beta]\end{matrix}\right]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^{(\alpha-1)}(r-1)^{(\alpha-1)}-1]}{(r-1)^{(\alpha-1)}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+\alpha-3)}{(\phi+\alpha-2)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix}[\alpha]\\ G_n^*\\ [a,r,s_1,\text{---},s_\beta]\end{matrix}$$

$$[S]_n \left[\begin{matrix}[\alpha]\\ G_n^*\\ [a,r,s_1,\text{---},s_\beta]\end{matrix}\right] = a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^{\alpha}(r-1)^{\alpha}-1]}{(r-1)^{\alpha}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+\alpha-2)}{(\phi+\alpha-1)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix}[\alpha+1]\\ G_n^*\\ [a,r,s_1,\text{---},s_\beta]\end{matrix}$$

$$[S]_n \left[\begin{matrix}[\alpha+1]\\ G_n^*\\ [a,r,s_1,\text{---},s_\beta]\end{matrix}\right]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^{(\alpha+1)}(r-1)^{(\alpha+1)}-1]}{(r-1)^{(\alpha+1)}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=\phi+1}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+\alpha-1)}{(\phi+\alpha)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [\alpha+2] \\ G_n^{*} \\ [a, r, s_1, \text{---}, s_\beta] \end{matrix}$$

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4] GENERALIZED GEOMETRIC PROGRESSIONS–RANDOM [$*^r$ GEOMETRIC PROGRESSIONS]

We define the

GENERALIZED GEOMETRIC PROGRESSIONS–RANDOM [$*^r$ GEOMETRIC PROGRESSION]

$$\begin{bmatrix} G_n^{*^r} \\ [a, r, s_1, \text{---}, s_\beta] \end{bmatrix}_{[@, 2, j_1, \text{---}, j_\beta]} = ar^{(n-1)} s_1^{\left(\frac{(n-j_1)(n-j_1+1)}{2!}\right)} \sum_{i=j_2}^{(n-1)} (n-i) s_2^{\left(\frac{(n-i)(n-i+1)}{2!}\right)}$$

$$\sum_{i=j_3}^{(n-1)} \left(\frac{(n-i)(n-i+1)}{2!}\right) s_3^{\left(\frac{(n-i)(n-i+1)}{2!}\right)} \text{------}$$

$$\sum_{i=j_\beta}^{(n-1)} \left(\frac{(n-i)(n-i+1)\text{---}(n-i+\beta-2)}{(\beta-1)!}\right) s_\beta^{\left(\frac{(n-i)(n-i+1)}{2!}\right)}$$

$$= ar^{(n-1)} \sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \left(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi-2)}{(\phi-1)!} s_\phi^{\left(\frac{(n-i)(n-i+1)}{2!}\right)}\right.$$

$$= \begin{matrix} [1] \\ G_n^{*^r} \\ [a, r, s_1, \text{---}, s_\beta] \\ [@, 2, j_1, \text{---}, j_\beta] \end{matrix} \qquad [n \geq 1]$$

"a" as the initiating Term, "r" as the Common Ratio and "s_1", "s_2", "s_3", --- , "s_β" as the inductive Ratios [$a, r, s_1, s_2, s_3, \text{---}, s_\beta$ are algebraic number $r \neq 1$] induced at [@] "j_1", "j_2", "j_3", ---, "j_β" respectively [$j_1 > 1, i = 1, 2\text{---}\beta$].

We can determine the Super–Sums of the GENERALIZED GEOMETRIC PROGRESSION– RANDOM up to any Level $\alpha = 1, 2, 3\text{---}\infty$. The ordinary Sum of GENERALIZED GEOMETRIC PROGRESSION–RANDOM is clearly the Super–Sum Level -1 of the same. "β" may be called Generalization–Level of the GENERALIZED GEOMETRIC PROGRESSION –RANDOM.

<u>Notation</u>

In $[S]_n \left[\begin{matrix}[\alpha]\\ G_n^{*^r}\\ [a, r, s_1, \text{---}, s_\beta]\end{matrix}\right]_{[@, s, j_1, \text{---}, j_\beta]}$, $[S]_n$ is the Sum upto n terms of the α^{th} Level of the GENERALIZED GEOMETRIC PROGRESSION – RANDOM [Super–Sum α^{th} Level] and it yields the $\begin{matrix}[\alpha+1]\\ G_n^{*^r}\\ [a, r, s_1, \text{---}, s_\beta]\\ [@, 2, j_1, \text{---}, j_\beta]\end{matrix}$ the $(\alpha+1)^{th}$ Level of the given GENERALIZED GEOMETRIC PROGRESSION – RANDOM.

$$\left[\begin{matrix}G_n^{*^r}\\ [a, r, s_1, \text{---}, s_\beta]\end{matrix}\right]_{[@, 2, j_1, \text{---}, j_\beta]}$$

$$= ar^{(n-1)} \sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \left(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi-2)}{(\phi-1)!}\right) s_\phi^{\left(\frac{(n-i)(n-i+1)}{2!}\right)}$$

$$= \begin{matrix}[1]\\ G_n^{*^r}\\ [a, r, s_1, \text{---}, s_\beta]\\ [@, 2, j_1, \text{---}, j_\beta]\end{matrix}$$

ALL the following Sequence of Formulae could be easily Proved by the Method of Mathematical Induction by now traditionally formalized. For each Level, the induction is performed on "n" and for The General Result the induction is performed on "α". The Routine Steps are omitted to save Eternal Space–Time !

$$[S]_n \begin{bmatrix} [1] \\ {G_n^*}^r \\ [a, r, s_1, \text{- - -}, s_\beta] \end{bmatrix} =$$
$$[@, 2, j_1, \text{- - -}, j_\beta]$$

$$\frac{a(r^n - 1)}{(r-1)} \sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{- - -}(n-i+\phi-1)}{\phi!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [2] \\ {G_n^*}^r \\ [a, r, s_1, \text{- - -}, s_\beta] \\ [@, 2, j_1, \text{- - -}, j_\beta] \end{matrix}$$

$$[S]_n \begin{bmatrix} [2] \\ {G_n^*}^r \\ [a, r, s_1, \text{- - -}, s_\beta] \end{bmatrix} =$$
$$[@, 2, j_1, \text{- - -}, j_\beta]$$

$$a\Big[\frac{r^{2(n+1)}[n(r-1)-1] - [n^2(r-1)^2 - 1]}{(r-1)^2[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \Big(\frac{(n-i)(n-i+1)\text{- - -}(n-i+\phi)}{(\phi+1)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [3] \\ {G_n^*}^r \\ [a, r, s_1, \text{- - -}, s_\beta] \\ [@, 2, j_1, \text{- - -}, j_\beta] \end{matrix}$$

$$[S]_n \begin{bmatrix} [3] \\ {G_n^*}^r \\ [a, r, s_1, \text{- - -}, s_\beta] \end{bmatrix} =$$
$$[@, 2, j_1, \text{- - -}, j_\beta]$$

$$a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^3(r-1)^3-1]}{(r-1)^3[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=j_\phi}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+1)}{(\phi+2)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{array}{c}[4]\\ G_n^{*r}\\ [a,r,s_1,\text{---},s_\beta]\\ [@,2,j_1,\text{---},j_\beta]\end{array}$$

$$[S]_n\left[\begin{array}{c}[4]\\ G_n^{*r}\\ [a,r,s_1,\text{---},s_\beta]\end{array}\right]_{[@,2,j_1,\text{---},j_\beta]} = a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^4(r-1)^4-1]}{(r-1)^4[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=j_\phi}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+2)}{(\phi+3)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{array}{c}[5]\\ G_n^{*r}\\ [a,r,s_1,\text{---},s_\beta]\\ [@,2,j_1,\text{---},j_\beta]\end{array}$$

$$[S]_n\left[\begin{array}{c}[5]\\ G_n^{*r}\\ [a,r,s_1,\text{---},s_\beta]\end{array}\right]_{[@,2,j_1,\text{---},j_\beta]} = a\Big[\frac{r^{(n+1)}[n(r-1)-1]-[n^5(r-1)^5-1]}{(r-1)^5[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta}\sum_{i=j_\phi}^{(n-1)}\Big(\frac{(n-i)(n-i+1)\text{---}(n-i+\phi+3)}{(\phi+4)!}\Big)s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [6] \\ G_n^{*^r} \\ [a, r, s_1, ---, s_\beta] \\ [@, 2, j_1, ---, j_\beta] \end{matrix}$$

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$$[S]_n \begin{bmatrix} [\alpha - 1] \\ G_n^{*^r} \\ [a, r, s_1, ---, s_\beta] \end{bmatrix} \atop [@, 2, j_1, ---, j_\beta]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{(\alpha-1)}(r-1)^{(\alpha-1)} - 1]}{(r-1)^{(\alpha-1)}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \Big(\frac{(n-i)(n-i+1)---(n-i+\phi+\alpha-3)}{(\phi+\alpha-2)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [\alpha] \\ G_n^{*^r} \\ [a, r, s_1, ---, s_\beta] \\ [@, 2, j_1, ---, j_\beta] \end{matrix}$$

$$[S]_n \begin{bmatrix} [\alpha] \\ G_n^{*^r} \\ [a, r, s_1, ---, s_\beta] \end{bmatrix} \atop [@, 2, j_1, ---, j_\beta] \; = \; a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{\alpha}(r-1)^{\alpha} - 1]}{(r-1)^{\alpha}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \Big(\frac{(n-i)(n-i+1)---(n-i+\phi+\alpha-2)}{(\phi+\alpha-1)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [\alpha + 1] \\ G_n^{*^r} \\ [a, r, s_1, ---, s_\beta] \\ [@, 2, j_1, ---, j_\beta] \end{matrix}$$

$$[S]_n \left[\begin{matrix} [\alpha+1] \\ {G_n^*}^r \\ [a, r, s_1, ---, s_\beta] \end{matrix} \right] \\ [@, 2, j_1, ---, j_\beta]$$

$$= a\Big[\frac{r^{(n+1)}[n(r-1)-1] - [n^{(\alpha+1)}(r-1)^{(\alpha+1)}-1]}{(r-1)^{(\alpha+1)}[n(r-1)-1]}\Big]$$

$$\sum_{\phi=1}^{\beta} \sum_{i=j_\phi}^{(n-1)} \Big(\frac{(n-i)(n-i+1) --- (n-i+\phi+\alpha-1)}{(\phi+\alpha)!}\Big) s_\phi^{\Big(\frac{(n-i)(n-i+1)}{2!}\Big)}$$

$$= \begin{matrix} [\alpha+2] \\ {G_n^*}^r \\ [a, r, s_1, ---, s_\beta] \\ [@, 2, j_1, ---, j_\beta] \end{matrix}$$

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CRAZY–GEOMETRIC PROGRESSIONS

The following possibilities may be noted.

$$\begin{bmatrix} CG_n^* \\ [a, r_1, r_2, \text{---}, r_\beta] \end{bmatrix} = a r_1^{(n-1)} r_2^{\left(\frac{(n-1)(n)}{2!}\right)} r_3^{\left(\frac{(n-1)(n)(n+1)}{3!}\right)} \text{---}$$

$$r_\beta^{\left(\frac{(n-1)(n)(n+1)\text{---}(n+\beta-2)}{\beta!}\right)} \qquad [n \geq 1]$$

"a" as the Initiating Term, "r_1", "r_2", "r_3", ---, "r_β" as the Inductive Ratios [a, r_1, r_2, r_3, ---, r_β are algebraic numbers $r_1 \neq 1$].

We may call this type of Progression THE CRAZY–GEOMETRIC PROGRESSSION. Analytic Formulae for the Super–Sums do not allow for any possible simplification in these cases. [See. 1]

We could of course induct and repeat the same inductive–block at as many random points of entry and generalize appropriately. The details though trivial are cumbersome. [See. 1]

Since the initiating term and Common Ratio and the Inductive Ratios can be any algebraic number, we can see that each family of the Progressions elucidated here, defines a unique Algebraic Field of Sequences.

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Bibliography

1] Narayanan Raghunathan :
Functions and their Progressions – An Elementary Text. [*unpublished*]