OM SHRI MAHA GANAPATHAYE NAMA:

OM POORNAMADA: POORNAMIDAM
POORNAT POORNAMUDACHYATE
POORNASYA POORNAMADAYA
POORNAMEVAVASHISHYATE

OM SHOONYAMADA: SHOONYAMIDAM
SHOONYAT SHOONYAMUDACHYATE
SHOONYASYA SHOONYAMADAYA
SHOONYAMEVAVASHISHYATE

RECTANGLES, GOLDEN RECTANGLE - A BRIEF NOTE

[Dedicated to Sri. CKUK - GPMHS - Palghat.]

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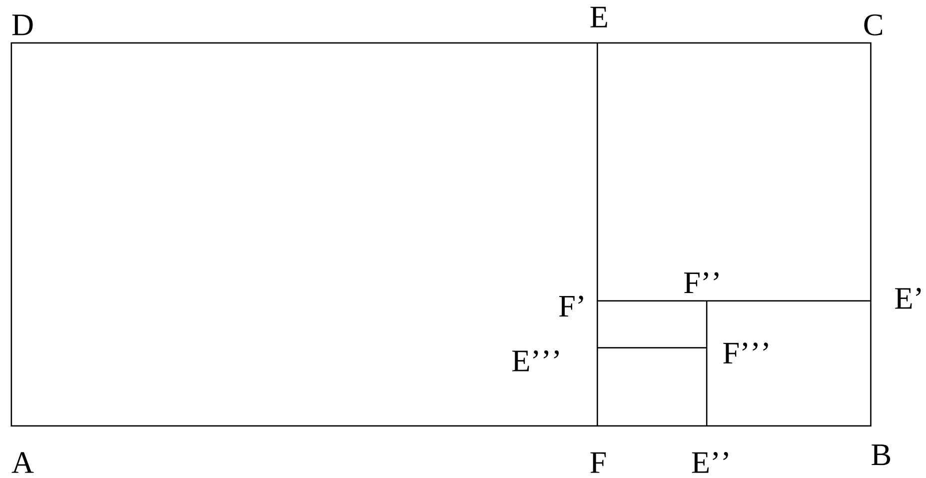
Abstract:
In a Golden Rectangle (known since antiquity), if the largest square is cut away, then the figure remaining will also be a Golden Rectangle. Such Rectangles are characterized by a length-breadth ratio $\frac{1+\sqrt{5}}{2}$, the Golden Ratio. This process clearly can be endlessly repeated. In this paper we note that for every Rectangle with a specific length-breadth ratio we can cut away a Rectangle with another characteristic length-breadth ratio such that the remaining Rectangle will have the same length-breadth ratio as the original ratio. This endlessly repeatable process clearly endows every Rectangle with a unique Extractable Sub-Rectangle (with a characteristic length-breadth ratio)

associated with it. The formula to determine the length-breadth ratio of the Extractable Sub-rectangle from the length-breadth ratio of the original Rectangle and vice-versa are derived here. The Sum of the Areas of the Infinite Sequence of Rectangles of the same length-breadth ratio as the original Rectangle is also derived. The Golden Rectangle clearly is a special case when the extracted Sub Rectangle is a square!

INFINITE RECTANGLES AND THEIR RAGHUNATHAN RATIOS [$\overline{\square}$]

We are clearly in the Euclidean-Terrain of Geometry. The ancient *terra-firma*! Let $ABCD$ be any Rectangle.



EVERY RECTANGLE is clearly characterized by the Fundamental Property that identifies it. ie. Its Length-Breadth Ratio [Basic Rectangular-Ratio].

$$\square = \frac{L}{B} = \frac{AB}{BC} > 1 \qquad [1.1]$$

We are going to prove that we can extract INFINITE Rectangles of the same Length - Breadth Ratio from any arbitrary Rectangle.

We want $\frac{AB}{BC} = \frac{BC}{CE}$. Let $CE = k_1$. ie. $\frac{L}{B} = \frac{B}{k_1}$

So we have $k_1 = \frac{B^2}{L}$.

Proceeding on inductively we can prove similarly that

$$k_1 = \frac{B^2}{L},\ k_2 = \frac{B^3}{L^2},\ k_3 = \frac{B^4}{L^3},\ ---,\ k_\tau = \frac{B^{(\tau+1)}}{L^\tau} \qquad [1.2]$$

$k_1 = CE$, $k_2 = BE'$, $k_3 = FE''$------∞ , The Length - Breadth Ratio of the Rectangle to be extracted from the original Rectangle is

$$\frac{AF}{AD} = \frac{(L - k_1)}{B} = \frac{(L - \frac{B^2}{L})}{B} = \frac{(L^2 - B^2)}{LB}$$

Dividing both the numerator and the denominator by B^2 and substituting

$\Box = \frac{L}{B}$ we have $\frac{(L^2 - B^2)}{LB} = \frac{(\Box^2 - 1)}{\Box}$

For EVERY RECTANGLE we can clearly SEE that the AREA of the RECTANGLE to be EXTRACTED from it to Retain Another RECTANGLE of the same RECTANGULAR RATIO [$\Box$] has the RECTANGULAR RATIO given by the FORMULA

$$\overline{\Box} = \frac{(L^2 - B^2)}{LB} = \frac{(L + B)(L - B)}{LB} = \frac{(\Box^2 - 1)}{\Box}$$
$$= \frac{(\Box + 1)(\Box - 1)}{\Box} \qquad [1.3]$$

We will call $\overline{\Box}$ THE RAGHUNATHAN RATIO of the RECTANGLE to HONOUR my FATHER.
When THE RAGHUNATHAN RATIO is 1 ie.

$\overline{\Box} = \frac{(L^2 - B^2)}{LB} = 1$ we have the Special Case of the so called Golden Rectangle when the RECTANGLE EXTRACTED is a SQUARE !

2] FORMULA TO EXTRACT THE BASIC RECTANGULAR RATIO FROM THE RAGHUNATHAN RATIO

Let

$$\overline{\Box} = \frac{(L^2 - B^2)}{LB}$$

Let $B = 1$ we have,

$$\frac{(L^2 - 1^2)}{L} = \overline{\square}$$

$$L^2 - 1 = \overline{\square} L$$

$$L^2 - \overline{\square} L - 1 = 0 \qquad [2.1]$$

Solving the quadratic equation and taking the positive value of the solution we have

$$L = \frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}$$

$$\square = \frac{L}{B} = \frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2} \qquad [2.2]$$

We will call this formula to determine the BASIC RECTANGULAR RATIO from the RAGHUNATHAN RATIO of THE RECTANGLE, THE RAJALAKSHMI FORMULA to HONOUR my MOTHER. When we substitutes $\overline{\square} = 1$ we have the Golden Rectangular Ratio

$\square = \dfrac{1 + \sqrt{5}}{2}$. (Ref: 1]-front coverpage. - 2]-p.136)

$1 < \square < \dfrac{1 + \sqrt{5}}{2} \iff 0 < \overline{\square} < 1$ and $\square > \dfrac{1 + \sqrt{5}}{2} \iff \overline{\square} > 1$

3] FORMULA FOR THE SUM OF THE AREAS OF THE INFINITE SEQUENCE OF RECTANGLES GENERATED WITH THE SAME RECTANGULAR RATIO

Here we derive a formula for the SUM OF THE AREAS OF INFINITE SEQUENCE OF RECTANGLES GENERATED WITH THE SAME RECTANGULAR RATIO.

Let $\{ \overset{\square}{A}_i \}$ be the AREAS of the Rectangles

$\overset{\square}{A}_1[ABCD] = LB$, $\overset{\square}{A}_2[FBCE] = \frac{B^3}{L}$, $\overset{\square}{A}_3[EBE'F'] = \frac{B^5}{L^3}$ - - - - - -∞

and let $[\overset{\square}{A}]$ denote the Sum of the Infinite Sequence of Areas.

$$[\overset{\square}{A}] = \sum_{i=1}^{\infty} \overset{\square}{A}_i = LB + \frac{B^3}{L} + \frac{B^5}{L^3} + \frac{B^7}{L^5} + \text{- - -} \qquad [3.1]$$

$$[\overset{\square}{A}] = LB + \frac{B^3}{L}\left[1 + \frac{B^2}{L^2} + \frac{B^4}{L^4} + \text{- - - - - -}\right]$$

Simplifying the Infinite Geometric Progression with the Common Ratio

$\frac{B^2}{L^2} < 1$, we have

$$[\overset{\square}{A}] = LB + \frac{B^3}{L}\left[\frac{1}{1 - \frac{B^2}{L^2}}\right]$$

$$[\overset{\square}{A}] = LB + \frac{B^3}{L}\left[\frac{L^2}{(L^2 - B^2)}\right] = LB + \left[\frac{LB^3}{(L^2 - B^2)}\right]$$

$$[\overset{\square}{A}] = LB\left[1+\frac{B^2}{(L^2-B^2)}\right] = \frac{LB(L^2)}{(L^2-B^2)} = \frac{(L^3B)}{(L^2-B^2)}$$

$$= \frac{L^3B}{(L+B)(L-B)} \qquad [3.2]$$

$$[\overset{\square}{A}] = \frac{LB(L^2)}{(L^2-B^2)} = \frac{L^2}{\square}\left[\frac{\square^2}{(\square^2-1)}\right] = \frac{L^2}{\square}\left[\frac{\square^2}{(\square+1)(\square-1)}\right] \qquad [3.3]$$

$$[\overset{\square}{A}] = \frac{L^2}{\square}\left[\frac{\square^2}{(\square+1)(\square-1)}\right] = L^2\left[\frac{\square}{(\square+1)(\square-1)}\right] \qquad [3.4]$$

$$[\overset{\square}{A}] = \frac{LB(L^2)}{(L^2-B^2)} = B^2\square\left[\frac{\square^2}{(\square^2-1)}\right] = B^2\square\left[\frac{\square^2}{(\square+1)(\square-1)}\right] \qquad [3.5]$$

We may also express $[\overset{\square}{A}]$ in terms of the Raghunathan Ratios. Substituting

$\square = \dfrac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}$ [2.2] in $[\overset{\square}{A}] = \dfrac{L^2}{\square}\left[\dfrac{\square^2}{(\square^2-1)}\right]$ [3.3] we have

$$[\overset{\square}{A}] = \left[\frac{L^2}{\left[\dfrac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]}\right]\left[\frac{\left[\dfrac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]^2}{\left[\dfrac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]^2-1}\right]$$

$$[\overset{\square}{A}] = \left[\frac{L^2}{\left[\dfrac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]}\right]\left[\frac{\dfrac{2\overline{\square}^2+2\overline{\square}\sqrt{\overline{\square}^2+4}+4}{4}}{\dfrac{2\overline{\square}^2+2\overline{\square}\sqrt{\overline{\square}^2+4}+4}{4}-1}\right]$$

$$[\overset{\square}{A}] = \left[\frac{L^2}{\left[\frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}\right]}\right] \left[\frac{2\overline{\square}^2 + 2\overline{\square}\sqrt{\overline{\square}^2 + 4} + 4}{2\overline{\square}^2 + 2\overline{\square}\sqrt{\overline{\square}^2 + 4}}\right]$$

$$[\overset{\square}{A}] = \left[\frac{L^2}{\left[\frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}\right]}\right] \left[\frac{\overline{\square}^2 + \overline{\square}\sqrt{\overline{\square}^2 + 4} + 2}{\overline{\square}^2 + \overline{\square}\sqrt{\overline{\square}^2 + 4}}\right] \quad [3.6]$$

Substituting

$\square = \frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}$ [2.2] in $[\overset{\square}{A}] = L^2 \left[\frac{\square}{(\square + 1)(\square - 1)}\right]$
[3.4] we have

$$[\overset{\square}{A}] = L^2 \left[\frac{\left[\frac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]}{\left[\frac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{2}\right]^2 - 1}\right] = L^2 \left[\frac{\overline{\square}+\sqrt{\overline{\square}^2+4}}{\overline{\square}^2+\overline{\square}\sqrt{\overline{\square}^2+4}}\right] \quad [3.7]$$

Substituting

$\square = \frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}$ [2.2] in $[\overset{\square}{A}] = B^2\square \left[\frac{\square^2}{(\square + 1)(\square - 1)}\right]$
[3.5] we have

$$[\overset{\square}{A}] = B^2 \left[\frac{\overline{\square} + \sqrt{\overline{\square}^2 + 4}}{2}\right] \left[\frac{\overline{\square}^2 + \overline{\square}\sqrt{\overline{\square}^2 + 4} + 2}{\overline{\square}^2 + \overline{\square}\sqrt{\overline{\square}^2 + 4}}\right] \quad [3.8]$$

The Formulae [3.2] - - - [3.8] may be called the Rajalakshmi-Raghunathan Formulae.

Sum of the Areas of the Rectangles Extracted.

We may now consider the Area of the Sum of the Rectangles Extracted. We can intuitively see that it must be the Area of the Rectangle $ABCD$ itself!

Let $\{ \overset{\overline{\Box}}{A}_i \}$ be the AREAS of the Rectangles Extracted. ie.

$$\overset{\overline{\Box}}{A}_1[AFED] = (LB - \frac{B^3}{L}),\ \overset{\overline{\Box}}{A}_2[F'E'CE] = (\frac{B^3}{L} - \frac{B^5}{L^3}),$$

$$\overset{\overline{\Box}}{A}_3[E''BE'F''] = (\frac{B^5}{L^3} - \frac{B^7}{L^5})\text{------}\infty$$

and let $[\overset{\overline{\Box}}{A}]$ denote the Sum of the Infinite Sequence of Extracted Areas.

$$[\overset{\overline{\Box}}{A}] = \sum_{i=1}^{\infty} \overset{\overline{\Box}}{A}_i = (LB - \frac{B^3}{L}) + (\frac{B^3}{L} - \frac{B^5}{L^3}) + (\frac{B^5}{L^3} - \frac{B^7}{L^5}) + \text{---}$$

$$[\overset{\overline{\Box}}{A}] = LB + (\frac{B^3}{L} + \frac{B^5}{L^3} + \frac{B^7}{L^5} + \text{---}) - (\frac{B^3}{L} + \frac{B^5}{L^3} + \frac{B^7}{L^5} + \text{---})$$

$$[\overset{\overline{\Box}}{A}] = LB \qquad [3.9]$$

Thus we may state an elegant (albeit trivial) Theorem

Every Rectangle with the Rectangular Ratio $\Box = \frac{L}{B}$ may be Decomposed into an Infinite Family of Rectangles [Raghunathan-Rectangles] with the Rectangular Ratio $\overline{\Box} = \frac{(L+B)(L-B)}{LB} = \frac{(\Box+1)(\Box-1)}{\Box}$

We will conclude this paper with the observation that for every Rectangle with a certain Basic Rectangular Ratio there exists a unique Logarithmic Spiral associated with it. The Infinite Family of Raghunathan Rectangles of every Rectangle Circumscribe a Unique Logarithmic Spiral. When the Circumscribing Rectangle Finally Becomes A Point, the Spiral Culminates into the Source! Naturally!

I thank my brother Anand and friend Adrian for sharing my joy. I hesitantly told them, “Perhaps Pythagores, Heraclitus (!) or Aryabhatta has already written my paper milleniums ago”. I apologize for my audacity of rescribing them, if these elementary results are already well-known!

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