Matrix Properties of The Most Generalized Fibonacci Sequences

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Abstract. By using matrix properties of the most generalized Fibonacci sequences $\{o_n\}$, some properties and quadratic relations of the sequences $\{o_n\}$ are obtained; the sequences are defined by the General Infinite Family of recurrence relations:

$$o_n = o_{n-1} + o_{n-2} + o_{n-3} + \ldots + o_{n-(\zeta-1)} + o_{n-\zeta}, n \geq \zeta$$
$$[\zeta = 2, 3, \ldots \alpha].$$

1 Inroduction

The Fibonacci sequence $\{F_n\}$ is defined as [1] $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$. This sequence has been generalized in different ways.

(a) Modifying the recurrence relations such that each term is the sum of three preceding terms [2] i.e.

$$P_n = P_{n-1} + P_{n-2} + P_{n-3}$$

where P_0, P_1, P_2 are arbitrary algebraic integers all of which are not zero.

(b) Other modifying recurrence relations such that each term is the sum of four preceding terms [3], i.e.

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4}$$

where Q_0, Q_1, Q_2, Q_3 are arbitrary algebraic integers all of which are not zero.

(c) Other modifying recurrence relations such that each term is the sum of five preceding terms [4], i.e.

$$D_n = D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4} + D_{n-5}, \quad n \geq 5$$

where D_0, D_1, D_2, D_3, D_4 are arbitrary algebraic integers all of which are not zero.

Now, we shall generalize with the infinite family of recurrence relations in which each term is the sum of $\zeta(xi)$ $[\zeta = 2, 3, \ldots \alpha]$ preceding terms.

2 The Generalized Sequences

$\{o_n\}$: We consider sequences:

$$\left\{o_n\right\} \equiv o_0,\ o_1,\ o_2, \ldots, o_n \ldots$$

where $o_0,\ o_1,\ o_2,\ o_3, \ldots, o_{\zeta-1}$, are arbitrary algebraic integers all of which are not zero and

$$\begin{aligned} & o_n = o_{n-1} + o_{n-2} + o_{n-3} + \ldots + o_{n-(\zeta-1)} + o_{n-\zeta}, n \geq \zeta \\ & [\zeta = 2, 3, \ldots \alpha]. \end{aligned} \tag{2.1}$$

For $\zeta = 2$, it gives the basic family of Fibonacci sequences and for $o_0 = 0$, $o_1 = 1$, it gives the Fundamental Fibonacci sequence and the arguments of this paper are superfluous. For $\zeta = 3, 4, \ldots \alpha$, it is valid. Whenever, inductive arguments are used for the proof, there are two steps involved. First, prove it for $\zeta = 3, 4$ and 5 in each case proving by induction ! [$\zeta = 3$ is enough]. We assume that the concerned result is true for $\zeta = 3, 4, 5, \ldots, k$ and close the doubly inductive argument by proving it for $\zeta = k + 1$. The cumbersome and self-evident steps are omitted. { For, $\zeta = 5$ we get the results of Sanjay K. Harne and C. L. Parihar [4]}.

We also consider the sequences

$$\left\{{}^{1}\Delta_n\right\} \equiv {}^{1}\Delta_0, {}^{1}\Delta_1, {}^{1}\Delta_2, \ldots {}^{1}\Delta_n \ldots$$

where ${}^{1}\Delta_0 = o_2 - o_1 - o_0$, ${}^{1}\Delta_1 = o_3 - o_2 - o_1$, ${}^{1}\Delta_2 = o_4 - o_3 - o_2, \ldots$

$${}^{1}\Delta_{(\zeta-2)} = o_\zeta - o_{\zeta-1} - o_{\zeta-2} \tag{2.2}$$

with

$$\begin{aligned} {}^{1}\Delta_n = o_{n-1} + o_{n-2} + o_{\zeta-2} + \ldots \\ + o_{n-(\zeta-2)} + o_{n-(\zeta-1)} \quad n \geq \zeta - 1 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \left\{{}^{2}\Delta_n\right\} \equiv {}^{2}\Delta_0, {}^{2}\Delta_1, {}^{2}\Delta_2, \ldots {}^{2}\Delta_n \ldots \\ {}^{2}\Delta_0 = n_3 - o_2 - o_1 - o_0,\ {}^{2}\Delta_1 = n_4 - o_3 - o_2 - o_1, \\ {}^{2}\Delta_2 = n_5 - o_4 - o_3 - o_2, \\ {}^{2}\Delta_{(\zeta-3)} = n_\zeta - o_{\zeta-1} - o_{\zeta-2} - o_{\zeta-3} \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} {}^{2}\Delta_n = o_{n-1} + o_{n-2} + \ldots \\ + o_{n-(\zeta-3)} + o_{n-(\zeta-2)} n \geq \zeta - 2 \ldots \end{aligned} \tag{2.5}$$

$$\left\{{}^{\zeta-3}\Delta_n\right\} \equiv {}^{\zeta-3}\Delta_0, {}^{\zeta-3}\Delta_1, {}^{\zeta-3}\Delta_2, \ldots {}^{\zeta-3}\Delta_n \ldots$$

where

$$
\begin{aligned}
{}^{\zeta-3}\Delta_0 &= o_{\zeta-2} - o_{\zeta-3} - \ldots - o_1 - o_0 \\
{}^{\zeta-3}\Delta_1 &= o_{\zeta-1} - o_{\zeta-2} - \ldots - o_2 - o_1 \\
{}^{\zeta-3}\Delta_2 &= o_{\zeta} - o_{\zeta-1} - \ldots - o_3 - o_2
\end{aligned} \tag{2.6}
$$

with

$$
{}^{\zeta-3}\Delta_n = o_{n-1} + o_{n-2} + o_{n-3} \quad n \geq 3
$$

and

$$
\begin{aligned}
\left\{{}^{\zeta-2}\Delta_n\right\} &\equiv {}^{\zeta-2}\Delta_0, {}^{\zeta-2}\Delta_1, {}^{\zeta-2}\Delta_2, \ldots {}^{\zeta-2}\Delta_n \ldots \\
{}^{\zeta-2}\Delta_0 &= o_{\zeta-1} - o_{\zeta-2} - \ldots - o_1 - o_0 \\
{}^{\zeta-2}\Delta_1 &= o_{\zeta-2} - o_{\zeta-2} - \ldots - o_2 - o_1
\end{aligned} \tag{2.7}
$$

with

$$
{}^{\zeta-2}\Delta_0 = o_{n-1} + o_{n-2} \geq 2 \tag{2.8}
$$

From (2.3) and (2.1), wee have for $n \geq \zeta + (\zeta - 1)$

$$
\begin{aligned}
{}^{1}\Delta_n &= o_{n-(\zeta+1)} + o_{n-\zeta} + o_{n-(\zeta-1)} + \ldots + o_{n-3} + o_{n-2} \\
&= o_{n-(\zeta+2)} + o_{n-(\zeta+1)} + o_{n-\zeta} + \ldots + o_{n-4} + o_{n-3}
\end{aligned}
$$

$$
\begin{aligned}
&= o_{n-(\zeta+3)} + o_{n-(\zeta+2)} + o_{n-(\zeta+1)} + \ldots + o_{n-5} + o_{n-4} \ldots \\
&= o_{n-[\zeta+(\zeta-1)]} + o_{n-[\zeta+(\zeta-2)]} + o_{n-[\zeta+(\zeta-3)]} + \ldots \\
&\quad + o_{n-(\zeta+1)} + o_{n-\zeta} \\
{}^1\Delta_n &= {}^1\Delta_{n-1} + {}^1\Delta_{n-2} +^1 \Delta_{n-3} + \ldots +^1 \Delta_{n-(\zeta-1)} +^1 \Delta_{n-\zeta}.
\end{aligned}
$$

Using (2.3) and (2.2), we obtain:

$$
\begin{aligned}
{}^1\Delta_{\zeta+(\zeta-2)} &= \Big(o_{\zeta+1} + o_\zeta + o_{\zeta-1} + o_{\zeta-2}\Big) \\
&\quad + \Big(o_\zeta + o_{\zeta-1}2 + o_{\zeta-2} + o_{\zeta-3}\Big) \ldots \\
&\quad + \Big(o_4 + o_3 + o_2 + o_1\Big) + \Big(o_3 + o_2 + o_1 + o_0\Big) \\
&\quad + \Big(o_\zeta - o_{\zeta-1} - o_{\zeta-2}\Big) \\
{}^1\Delta_{\zeta+(\zeta-2)} &= {}^1\Delta_{\zeta+(\zeta-3)} + {}^1\Delta_{\zeta+(\zeta-4)} + \ldots \\
&\quad + {}^1\Delta_\zeta + {}^1\Delta_{\zeta-1} + {}^1\Delta_{\zeta-2}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
{}^1\Delta_{\zeta+(\zeta-3)} &= {}^1\Delta_{\zeta+(\zeta-4)} + {}^1\Delta_{\zeta+(\zeta-5)} + \ldots \\
&\quad + {}^1\Delta_{\zeta-1} + {}^1\Delta_{\zeta-2} + {}^1\Delta_{\zeta-3} \\
{}^1\Delta_{\zeta+(\zeta-4)} &= {}^1\Delta_{\zeta+(\zeta-5)} + {}^1\Delta_{\zeta+(\zeta-6)} + \ldots \\
&\quad + {}^1\Delta_{\zeta-2} + {}^1\Delta_{\zeta-3} + {}^1\Delta_{\zeta-4} \\
{}^1\Delta_{\zeta+1} &= {}^1\Delta_\zeta +^1 \Delta_{\zeta-1} + \ldots + {}^1\Delta_3 +^1 \Delta_2 +^1 \Delta_1.
\end{aligned}
$$

Hence, we have for $n \geq \zeta$

$$^{1}\Delta_n = {}^{1}\Delta_{n-1} + {}^{1}\Delta_{n-2} +^{1} \Delta_{n-3} + \ldots +^{1} \Delta_{n-(\zeta-1)} +^{1} \Delta_{n-\zeta}. \quad (2.9)$$

Proceeding on similar lines, it can be easily shown that for $n \geq \zeta$.

$$\begin{aligned} {}^{2}\Delta_n &= o_{n-2} + o_{n-3} + \ldots + o_{n-\zeta} + o_{n-(\zeta+1)} \\ &\quad + o_{n-3} + o_{n-4} + \ldots + o_{n-(\zeta+1)} + o_{n-(\zeta+2)} \cdots \\ &\quad + o_{n-(\zeta-1)} + o_{n-\zeta} + \ldots \\ &\quad + o_{n-(\zeta+\zeta-3)} + o_{n-(\zeta+\zeta-2)} \\ {}^{2}\Delta_n &= {}^{2}\Delta_{n-1} + {}^{2}\Delta_{n-2} + \ldots \\ &\quad + {}^{2}\Delta_{n-(\zeta-1)} + {}^{2}\Delta_{n-\zeta} \text{ for } n \geq \zeta. \end{aligned} \quad (2.10)$$

Proceeding on similar lines, it can be easily shown that for $n \geq \zeta$ $[\zeta = 3, 4, 5, \ldots \alpha]$ and for ${}^{3}\Delta_n$, ${}^{4}\Delta_n, \ldots$ similar relations hold finally culminating in ${}^{\zeta-2}\Delta_n$ such that

$$\begin{aligned} {}^{\zeta-2}\Delta_n &= o_{n-2} + o_{n-3} + \ldots + o_{n-\zeta} + o_{n-(\zeta+1)} \\ &\quad + o_{n-3} + o_{n-4} + \ldots + o_{n-(\zeta+1)} + o_{n-(\zeta+2)} \\ {}^{\zeta-2}\Delta_n &= {}^{\zeta-2}\Delta_{n-1} + {}^{\zeta-2}\Delta_{n-2} + \ldots \\ &\quad + {}^{\zeta-2}\Delta_{n-(\zeta-1)} + {}^{\zeta-2}\Delta_{n-\zeta} \text{ for } n \geq \zeta. \end{aligned} \quad (2.11)$$

Thus, the $(\zeta - 2)$ sequences $\{{}^{3}\Delta_n\}$, $\{{}^{2}\Delta_n\}, \ldots \{{}^{\zeta-3}\Delta_n\}$ and $\{{}^{\zeta-2}\Delta_n\}$ are special cases for sequences $\{o_n\}$ and are obtained by taking different initial values.

$$o_n = o_{n-1} + o_{n-2} + o_{n-3} + \ldots + o_{n-(\zeta-1)} + o_{n-\zeta}, n \geq \zeta$$
$$[\zeta = 2, 3, \ldots \alpha]. \tag{2.1}$$

on taking

$$o_0 = o_1 = \ldots = o_{\zeta-4} = 0;\ o_{\zeta-3} = 1;\ o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = o_1 = \ldots = o_{\zeta-5} = 0;\ o_{\zeta-4} = 1;\ o_{\zeta-3} = 0;$$
$$o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = o_1 = \ldots = o_{\zeta-6} = 0;\ o_{\zeta-5} = 1;\ o_{\zeta-4} = o_{\zeta-3} = 0;$$
$$o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = o_1 = \ldots = o_{\zeta-7} = 0;\ o_{\zeta-6} = 1;\ o_{\zeta-5} = \ldots = o_{\zeta-3} = 0;$$
$$o_{\zeta-2} = 1;\ o_{\zeta-1} = 2 \ldots$$

$$o_0 = o_1 = 0;\ o_2 = 1;\ o_3 \ldots = o_{\zeta-3} = 0;\ o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = 0; o_1 = 1;\ o_2 = \ldots = o_{\zeta-3} = 0;\ o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = 1; o_1 = o_2 = \ldots = o_{\zeta-3} = 0;\ o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$o_0 = o_1 = o_2 = o_3 \ldots o_{\zeta-3} = 0;\ o_{\zeta-2} = 1;\ o_{\zeta-1} = 2$$

$$
\begin{aligned}
&0,0,\ldots 0,0,1,1,2,\ldots,{}^{1}A_n,\ldots\\
&0,0,\ldots 0,1,0,1,2,\ldots,{}^{2}A_n,\ldots\\
&0,0,\ldots 1,0,0,1,2,\ldots,{}^{3}A_n,\ldots\\
&0,0,\ldots 1,0,0,0,1,2,\ldots,{}^{4}A_n,\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&0,0,1,\ldots 0,1,2,\ldots,{}^{\zeta-4}A_n,\ldots\\
&0,1,\ldots 0,1,2,\ldots,{}^{\zeta-3}A_n,\ldots\\
&1,0,\ldots 0,1,2,\ldots,{}^{\zeta-2}A_n,\ldots\\
&0,0,\ldots 0,1,2,\ldots,{}^{\zeta-1}A_n,\ldots
\end{aligned}
\tag{2.12}
$$

Here, we find that

$$
\begin{aligned}
{}^{2}A_n &= {}^{1}A_{n-1} +^{1} A_{n-2} +^{1} A_{n-3} + \ldots {}^{1}A_{n-(\zeta-1)}\\
{}^{3}A_n &= {}^{1}A_{n-1} +^{1} A_{n-2} +^{1} A_{n-3} + \ldots {}^{1}A_{n-(\zeta-2)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
{}^{\zeta-2}A_n &= {}^{1}A_{n-1} +^{1} A_{n-2} +^{1} A_{n-3}\\
{}^{\zeta-1}A_n &= {}^{1}A_{n-1} +^{1} A_{n-2}.
\end{aligned}
$$

Thus, we can say that $({}^{1}\Delta_n)$ is a o_n-type sequence while $({}^{2}\Delta_n)$ is a $({}^{1}\Delta_n)$-type sequence etc…and $({}^{\zeta-2}\Delta_n)$ is a ${}^{\zeta-3}\Delta_n$-type sequence and $({}^{\zeta-1}\Delta_n)$ is a ${}^{\zeta-2}\Delta_n$-type sequence.

3 Simple properties

Let us consider the $[\zeta \times \zeta]$ matrices where $\zeta = 3, 4, 5, \ldots \alpha$

$$X \equiv \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}. \tag{3.1}$$

By repeated mathematical induction it is easily proved that:

$$[X]^n = \begin{bmatrix} {}^1A_{n+1} & {}^2A_{n+1} & \cdots & {}^{\zeta-2}A_{n+1} & {}^{\zeta-1}A_{n+1} & {}^1A_n \\ {}^1A_n & {}^2A_n & \cdots & {}^{\zeta-2}A_n & {}^{\zeta-1}A_n & {}^1A_{n-1} \\ {}^1A_{n-1} & {}^2A_{n-1} & \cdots & {}^{\zeta-2}A_{n-1} & {}^{\zeta-1}A_{n-1} & {}^1A_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ {}^1A_{n-(\zeta-3)} & {}^2A_{n-(\zeta-3)} & \cdots & {}^{\zeta-2}A_{n-(\zeta-3)} & {}^{\zeta-1}A_{n-(\zeta-3)} & {}^1A_{n-(\zeta-2)} \\ {}^1A_{n-(\zeta-2)} & {}^2A_{n-(\zeta-2)} & \cdots & {}^{\zeta-2}A_{n-(\zeta-2)} & {}^{\zeta-1}A_{n-(\zeta-2)} & {}^1A_{n-(\zeta-1)} \end{bmatrix}$$

$$n \geq \zeta - 1 \tag{3.2}$$

and

$$\Big[o_n, o_{n-1}, o_{n-2}, \ldots, o_{n-(\zeta-2)}, o_{n-(\zeta-1)}\Big] = X^{n-(\zeta-1)}\Big[o_{(\zeta-1)}, o_{(\zeta-2)}, \ldots, n_2, o_1, o_0\Big], \quad n \geq \zeta - 1. \quad (3.3)$$

On using (3.2) and (3.3) we get:

$$\begin{bmatrix} o_{n+p} \\ o_{n+p-1} \\ o_{n+p-2} \\ \cdots \\ o_{n+p-(\zeta-2)} \\ o_{n+p-(\zeta-1)} \end{bmatrix} = \begin{bmatrix} {}^1A_{n+1} & {}^2A_{n+1} & \cdots & {}^{\zeta-2}A_{n+1} & {}^{\zeta-1}A_{n+1} & {}^1A_n \\ {}^1A_n & {}^2A_n & \cdots & {}^{\zeta-2}A_n & {}^{\zeta-1}A_n & {}^1A_{n-1} \\ {}^1A_{n-1} & {}^2A_{n-1} & \cdots & {}^{\zeta-2}A_{n-1} & {}^{\zeta-1}A_{n-1} & {}^1A_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^1A_{n-(\zeta-3)} & {}^2A_{n-(\zeta-3)} & \cdots & {}^{\zeta-2}A_{n-(\zeta-3)} & {}^{\zeta-1}A_{n-(\zeta-3)} & {}^1A_{n-(\zeta-2)} \\ {}^1A_{n-(\zeta-2)} & {}^2A_{n-(\zeta-2)} & \cdots & {}^{\zeta-2}A_{n-(\zeta-2)} & {}^{\zeta-1}A_{n-(\zeta-2)} & {}^1A_{n-(\zeta-1)} \end{bmatrix} \begin{bmatrix} o_n \\ o_{n-1} \\ o_{n-2} \\ \cdots \\ o_{n-(\zeta-2)} \\ o_{n-(\zeta-1)} \end{bmatrix}.$$

From this we obtain:

$$o_{n+p} = {}^1A_{p+1}o_n + {}^2A_{p+1}o_{n-1} + \ldots + {}^{\zeta-2}A_{p+1}o_{n-(\zeta-3)} + {}^{\zeta-1}A_{p+1}o_{n-(\zeta-2)} +^1 A_n o_{n-(\zeta-1)}. \quad (3.4)$$

Let us consider the matrices $[Y]$ which are the transposes of the matrices

$[X]$, i.e.,

$$[Y] = [X] \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

It can be shown easily that the sequences:

$$o_3, {}^1\Delta_4, \dots {}^{\zeta-3}\Delta_4, {}^{\zeta-2}\Delta_4, o_4, \dots o_{n-1}, \\ {}^1\Delta_n, \dots {}^{\zeta-3}\Delta_n, {}^{\zeta-2}\Delta_n, o_n \quad (3.5)$$

are generalized by the matrices $[Y]$, that is,

$$\left[o_n, {}^1\Delta_n, \dots {}^{\zeta-3}\Delta_n, {}^{\zeta-2}\Delta_n, o_{n-1}\right] \\ = X^{n-(\zeta-1)}\left[o_{\zeta-1}, {}^1\Delta_{\zeta-1}, \dots {}^{\zeta-3}\Delta_{\zeta-1}, {}^{\zeta-2}\Delta_{\zeta-1}, o_{\zeta-2}\right] n \geq \zeta - 1. \quad (3.6)$$

On using (3.6) and (3.5), we get:

$$\left[o_{n+p}, {}^{1}\Delta_{n+p}, \ldots {}^{\zeta-3}\Delta_{n+p}, {}^{\zeta-2}\Delta_{n+p}, {}^{1}A_{n+p-1}\right]$$

$$= y^{n+p-(\eta-1)}\left[o_{\zeta-1}, {}^{1}\Delta_{\zeta-1}, \ldots {}^{\zeta-3}\Delta_{\zeta-1}, {}^{\zeta-2}\Delta_{\zeta-1}, o_{\zeta-2}\right]$$

$$= [Y]^{p}\left[o_{n}, {}^{1}\Delta_{n}, \ldots {}^{\zeta-3}\Delta_{n}, {}^{\zeta-2}\Delta_{n}, o_{\zeta-1}\right]$$

$$= \begin{bmatrix} {}^{1}A_{p+1} & {}^{1}A_{p} & {}^{1}A_{p-1} & \cdots & {}^{1}A_{p-(\zeta-3)} & {}^{1}A_{p-(\zeta-2)} \\ {}^{2}A_{p+1} & {}^{2}A_{p} & {}^{2}A_{p-1} & \cdots & {}^{2}A_{p-(\zeta-3)} & {}^{2}A_{p-(\zeta-2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^{\zeta-2}A_{p+1} & {}^{\zeta-2}A_{p} & {}^{\zeta-2}A_{p-1} & \cdots & {}^{\zeta-2}A_{p-(\zeta-3)} & {}^{\zeta-2}A_{p-(\zeta-2)} \\ {}^{\zeta-1}A_{p+1} & {}^{\zeta-1}A_{p} & {}^{\zeta-1}A_{p-1} & \cdots & {}^{\zeta-1}A_{p-(\zeta-3)} & {}^{\zeta-1}A_{p-(\zeta-2)} \\ {}^{1}A_{p} & {}^{1}A_{p-1} & {}^{1}A_{p-2} & \cdots & {}^{1}A_{p-(\zeta-2)} & {}^{1}A_{p-(\zeta-1)} \end{bmatrix} \begin{bmatrix} o_{n} \\ {}^{1}\Delta_{n} \\ \cdots \\ {}^{\zeta-3}\Delta_{n} \\ {}^{\zeta-2}\Delta_{n} \\ o_{n-1} \end{bmatrix}$$

Thus

$$\begin{aligned} o_{n+p} &= {}^{1}A_{p+1} + {}^{1}A_{p}{}^{1}\Delta_{n} + {}^{1}A_{p-1}{}^{2}\Delta_{n} \ldots \\ &\quad + {}^{1}A_{p-(\zeta-3)}{}^{\zeta-2}\Delta_{n} + {}^{1}A_{p-(\zeta-2)}o_{n-1} \\ {}^{1}\Delta_{n+p} &= {}^{2}A_{p+1} + {}^{2}A_{p}{}^{1}\Delta_{n} + {}^{2}A_{p-1}{}^{2}\Delta_{n} \ldots \\ &\quad + {}^{2}A_{p-(\zeta-3)}{}^{\zeta-2}\Delta_{n} + {}^{2}A_{p-(\zeta-2)}o_{n-1} \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
\begin{aligned}
{}^{\zeta-3}\Delta_{n+p} &= {}^{\zeta-2}A_{p+1}o_n + {}^{\zeta-2}A_p{}^1\Delta_n +^{\zeta-2} A_{p-1}{}^2\Delta_n \dots \\
&\quad + {}^{\zeta-2}A^{\zeta-2}_{p-(\zeta-3)}\Delta_n + {}^{\zeta-2}A_{p-(\zeta-2)}o_{n-1} \\
{}^{\zeta-2}\Delta_{n+p} &= {}^{\zeta-1}A_{p+1}o_n + {}^{\zeta-1}A_p{}^1\Delta_n +^{\zeta-1} A_{p-1}{}^2\Delta_n \dots \\
&\quad + {}^{\zeta-1}A^{\zeta-2}_{p-(\zeta-3)}\Delta_n + {}^{\zeta-1}A_{p-(\zeta-2)}o_{n-1}.
\end{aligned}
$$

4 Quadratic relations

It is easily by seen by Repeated Induction that for $n \geq (\zeta - 1)$

$$
\begin{aligned}
&\Big[\zeta = 3, 4, 5, \dots \alpha\Big] \\
&\Big[o_n, o_{n-1}, o_{n-2}, \dots, o_{n-(\zeta-2)}, o_{n-(\zeta-1)}\Big] \\
&\qquad = \Big[o_{(\zeta-1)}, o_{(\zeta-2)}, \dots, o_2, o_1, o_0\Big], \quad [Y]^{n-1}
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&\Big[o_n, {}^1\Delta_n, \dots {}^{\zeta-3}\Delta_n,^{\zeta-2}\Delta_n, o_{n-1}\Big] \\
&\qquad = \Big[o_{\zeta-1}, {}^1\Delta_{\zeta-1}, \dots {}^{\zeta-3}\Delta_{\zeta-1},^{\zeta-2}\Delta_{\zeta-1} o_{\zeta-2}\Big][X]^{n-(\zeta-1)}.
\end{aligned}
\tag{4.2}
$$

We shall now use the vector matrix representations of $o_r s$ to prove the following family of relations:

$$\begin{aligned} &o_n^2 + o_{n-1}^2 + o_{n-2}^2 + \ldots + o_{n-(\zeta-2)}^2 \\ &\qquad + 2o_{n-1}\Big\{o_{n-2} + \ldots + o_{n-(\zeta-2)} + o_{n-(\zeta-1)}\Big\} \\ &\quad = o_{\zeta-1}o_{2n-(\zeta-1)} + o_{\zeta-2}{}^1\Delta_{2n-(\zeta-1)} \cdots \\ &\qquad + o_2{}^{\zeta-3}\Delta_{2n-(\zeta-1)} + o_1{}^{\zeta-2}\Delta_{2n-(\zeta-1)} + o_0 o_{2n-\zeta} \end{aligned} \tag{4.3}$$

and also

$$\begin{aligned} &= o_{\zeta-1}o_{2n-(\zeta-1)} +^1 \Delta_{\zeta-1}o_{2n-\zeta} + \ldots +^{\zeta-3} \Delta_{\zeta-1}o_{2n-(2\zeta-4)} \\ &\qquad +^{\zeta-2} \Delta_{\zeta-1}o_{2n-(2\zeta-3)} + o_{\zeta-2}o_{2n-(2\zeta-2)}. \end{aligned} \tag{4.4}$$

Proof. The left hand sides are the scalar products of the vectors.

$$\Big[o_n, o_{n-1}, o_{n-2}, \ldots, o_{n-(\zeta-2)}, o_{n-(\zeta-1)}\Big]$$

and

$$\Big[o_n, {}^1\Delta_n, \ldots, {}^{\zeta-3}\Delta_n, {}^{\zeta-2}\Delta_n, o_{n-1}\Big].$$

Putting the value of equation (4.1) in the left hand side of equation (4.3), we get:

$$\begin{aligned} &\Big[o_{\zeta-1}, o_{n-2}, \ldots, o_2, o_1, o_0\Big] X^{n-4} \\ &\Big[o_n, {}^1\Delta_n, \ldots, {}^{\zeta-3}\Delta_n, {}^{\zeta-2}\Delta_n, o_{n-1}\Big]. \end{aligned}$$

Now, using (3.6) in the above equations, we get:

$$\Big[o_{\zeta-1}, o_{n-2}, \ldots, o_2, o_1, o_0\Big]$$
$$\Big[o_{2n-(\zeta-1)}, {}^1\Delta_{2n-(\zeta-1)}, \ldots, {}^{\zeta-3}\Delta_{2n-(\zeta-1)}, {}^{\zeta-2}\Delta_{2n-(\zeta-1)}, o_{2n-1}\Big]$$

which yields the R. H. S. of equation (4.3). The second set of relations (4.4) can also be similarly proved because:

$$\begin{aligned}
&\Big[o_n, {}^1\Delta_n, \ldots, {}^{\zeta-3}\Delta_n, {}^{\zeta-2}\Delta_n, o_{n-1}\Big] \\
&\qquad \Big[o_n, o_{n-1}, o_{n-2}, \ldots, o_{n-(\zeta-2)}, o_{n-(\zeta-1)}\Big] \\
&\quad = \Big[o_{\zeta-1}, {}^1\Delta_{\zeta-1}, \ldots, {}^{\zeta-3}\Delta_{\zeta-1}, {}^{\zeta-2}\Delta_{\zeta-1}, o_{\zeta-2}\Big] Y^{n-(\zeta-1)} \\
&\qquad \Big[o_n, o_{n-1}, o_{n-2}, \ldots, o_{n-(\zeta-2)}, o_{n-(\zeta-1)}\Big] \\
&\quad = \Big[o_{\zeta-1}, {}^1\Delta_{\zeta-1}, \ldots, {}^{\zeta-3}\Delta_{\zeta-1}, {}^{\zeta-2}\Delta_{\zeta-1}, o_{\zeta-2}\Big] \\
&\qquad \Big[o_{2n-(\zeta-1)}, o_{2n-\zeta}, \ldots, o_{2n-(2\zeta-4)}, o_{2n-(\zeta-3)}, o_{2n-(\zeta-2)}\Big] \\
&\quad = o_{\zeta-1}o_{2n-(\zeta-1)} + {}^1\Delta_{\zeta-1}o_{2n-\zeta} + \ldots + {}^{\zeta-3}\Delta_{\zeta-1}o_{2n-(2\zeta-4)} \\
&\qquad + {}^{\zeta-2}\Delta_{\zeta-1}o_{2n-(2\zeta-3)} + o_{\zeta-2}o_{2n-(2\zeta-2)} \\
&\quad = R.H.S.
\end{aligned}$$

□

Before I conclude, I must confess that it is the sheer Symmetry that leads me to this generalized paper. If we substitute $\zeta = 5$, we get Sanjay K. Harne and C. L. Parihar's results. We can clearly recognize now, that for each value of $\zeta \geq 3$ $[\zeta = 3, 4, \ldots, 6, 7, \ldots, \alpha]$, we get exactly similar symmetric results. Surely, strange is this vast magic of Fibonacci symmetry.

Dedication

This paper is dedicated in memory of my esteemed teacher Shri K. M. Nair, SSKZM.

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